Closing Tue: 14.4, 14.7
Closing Thu: 15.1, 15.2
Start on 14.7!!! Types of questions:

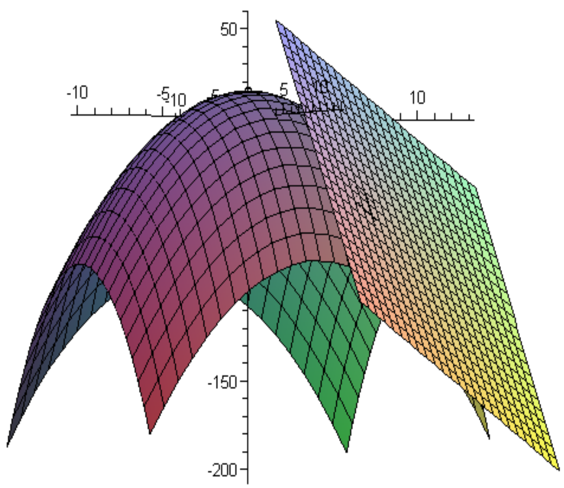
- Local max/min (HW 14.7/1-5)
- Global max/min (HW 14.7/6-8)
- Applied max/min (HW 14.7/9-14)

14.4 Tangent Planes (linear approx.)

The <u>tangent plane</u> to a surface at a point is the plane that contains all tangent lines at that point.

Example: $z = 15 - x^2 - y^2$ at

$$(x, y, z) = (7,4,-50)$$



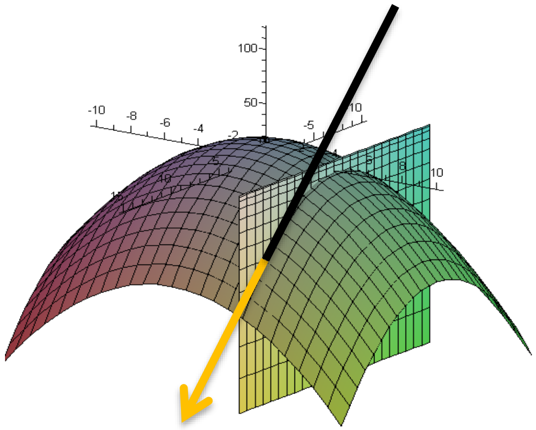
Derivation of Tangent Plane

The plane goes thru (7, 4, -50).
Now we need a normal vector.

Note:

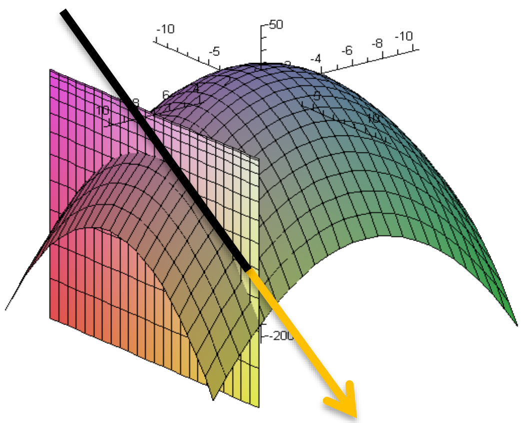
$f_x(x,y) = -2x$

$f_x(7,4) = -14$



$f_y(x,y) = -2y$

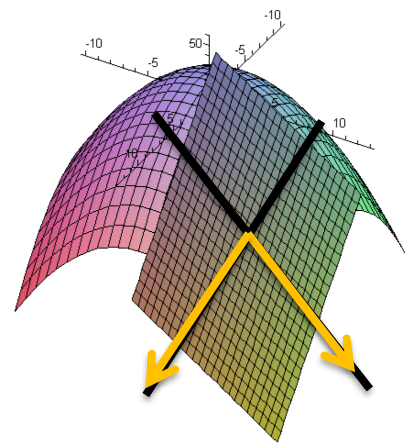
$f_y(7,4) = -8$



Thus, we can get two vectors that are parallel to the plane:

$$\langle 1,0,f_x(x_0,y_0)\rangle = \langle 1,0,-14\rangle$$
$$\langle 0,1,f_y(x_0,y_0)\rangle = \langle 0,1,-8\rangle$$

So a normal vector is given by
$\langle 1,0,-14 \rangle \times \langle 0,1,-8\rangle = \langle 14, 8, 1 \rangle$



Tangent Plane:

$$14(x-7) + 8(y-4) + (z+50) = 0$$

Which we rewrite as:

$$z + 50 = -14(x-7) - 8(y-4)$$

Aside: General Derivation

In general, for $z = f(x,y)$ at (x_0, y_0):

1. $z_0 = f(x_0, y_0)$ = height.

2. $\langle 1,0, f_x(x_0, y_0)\rangle$ = 'a tangent in *x*-dir.'
   $\langle 0,1, f_y(x_0, y_0)\rangle$ = 'a tangent in *y*-dir.'

3. Normal to surface:

$$\langle 1,0, f_x(x_0, y_0)\rangle \times \langle 0,1, f_y(x_0, y_0)\rangle = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$$

Tangent Plane:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which we typically write as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example:

Find the tangent plane for

$$f(x,y) = x^2 + 3y^2x - y^3$$

at (x,y) = (2,1).

Quick Application:

Use the ***linear approximation*** (or ***linearization*** or ***tangent plane approximation***) to

$$f(x,y) = x^2 + 3y^2x - y^3$$

at (x,y) = (2,1) to estimate the value of f(1.9, 1.05).

14.7 Max/Min

A **critical point** is a point (a,b) where

<u>BOTH</u>

$$f_x(a,b) = 0 \quad \text{AND} \quad f_y(a,b) = 0$$

or where either partial doesn't exist.

Example: Find the critical points of

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

Second Derivative Test

Let (a,b) be a critical point.

Compute

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

1. If $\boldsymbol{D} > \mathbf{0}$, (concavity SAME in all dir.)

   (a) If $\boldsymbol{f}_{\boldsymbol{xx}} > \mathbf{0}$ (concave UP all dir.)
   Local Minimum

   (b) If $\boldsymbol{f}_{\boldsymbol{xx}} < \mathbf{0}$ (concave DOWN all dir.)
   Local Maximum

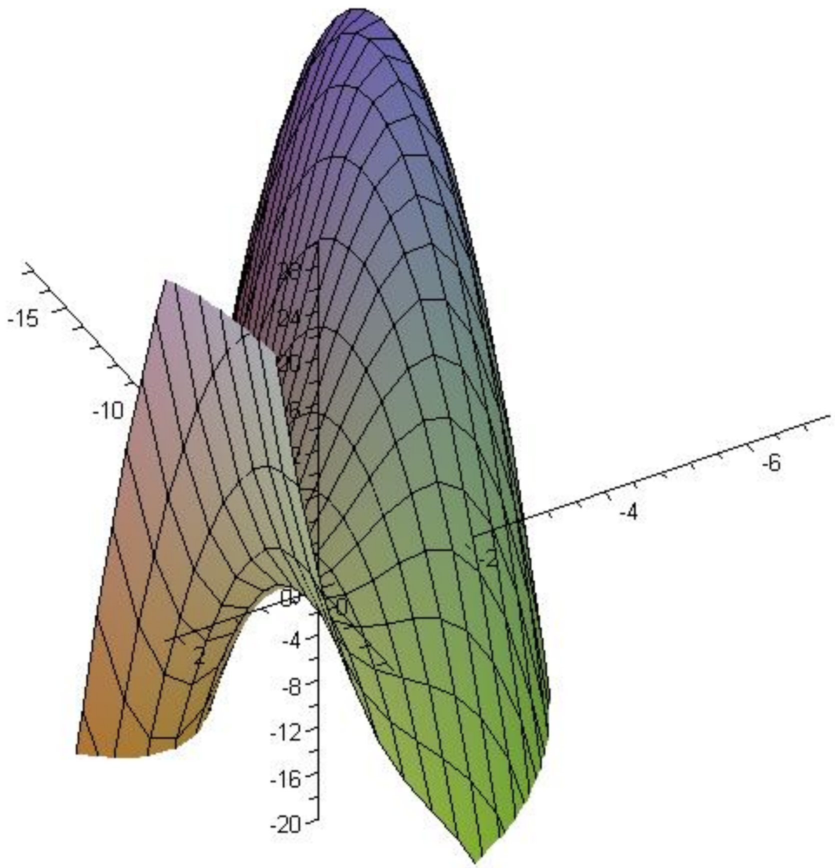
2. If **D < 0** (conc. CHANGES in *some* dir.)
   Saddle Point

3. If **D = 0**, the test is **inconclusive** .
   (need a contour map)

Example: (*same example*)

Find *and* classify all critical points for

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$



-15
-10
-6
-4
-2
-4
-8
-12
-16
-20

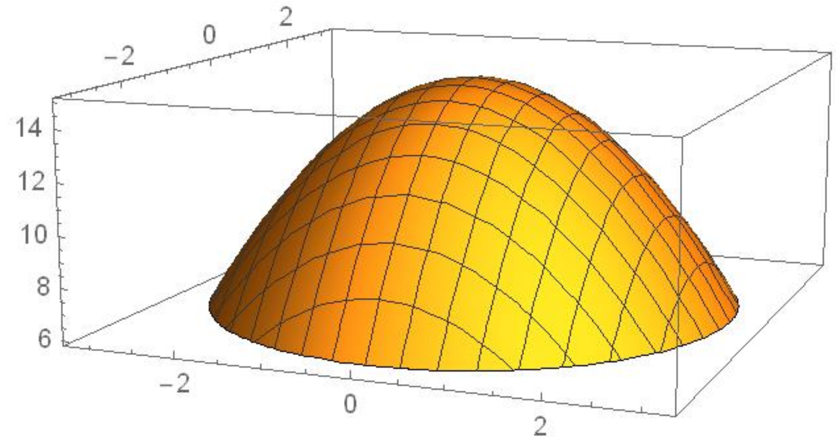
Quick Examples: All three examples have a critical point at (0,0).

1. $f(x,y) = 15 - x^2 - y^2$,

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$

$D = (-2)(-2)-(0)^2 = 4$

$D > 0$, $f_{xx} < 0$, $f_{yy} < 0$

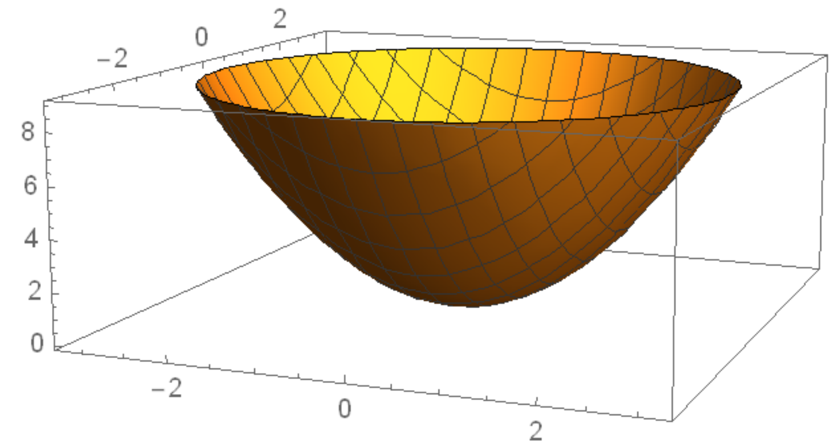


2. $f(x,y) = x^2 + y^2$,

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$,

$D = (2)(2)-(0)^2 = 4$

$D > 0$, $f_{xx} > 0$, $f_{yy} > 0$

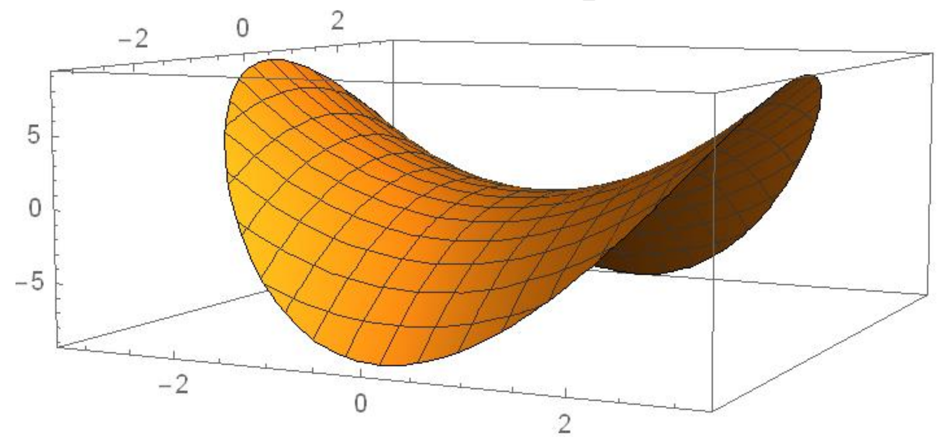


3. $f(x,y) = x^2 - y^2$

$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$,

$D = (2)(-2)-(0)^2 = -4$

$D < 0$ (note also, $f_{xx} < 0$, $f_{yy} > 0$)



Examples from old exams:

1. Find and classify all critical points for

$$f(x,y) = x^2 + 4y - x^2y + 1$$

2. Find and classify all critical points for

$$f(x,y) = \frac{9}{x} + 3xy - y^2$$

3. Find and classify all critical points for

$$f(x, y) = x^2y - 9y - xy^2 + y^3$$

Global Max/Min: Consider a surface z=f(x,y) over region R on the xy-plane. The **absolute/global max/min** over R are the largest/smallest z-values.

Key fact (Extreme value theorem)
The absolute max/min must occur at

1. A critical point, or
2. A boundary point.

How to do global max/min problems:
Step 1: Find critical pts inside region.
Step 2: Find critical numbers and corners above each boundary.
Step 3: Evaluate the function at all pts from steps 1 and 2.

Biggest output = global max
Smallest output = global min

Easy Example: Consider the paraboloid
$$z = x^2 + y^2 + 3$$
above the circular disk $x^2 + y^2 \leq 4$. Find the absolute max and min.

Boundaries (step 2) details:

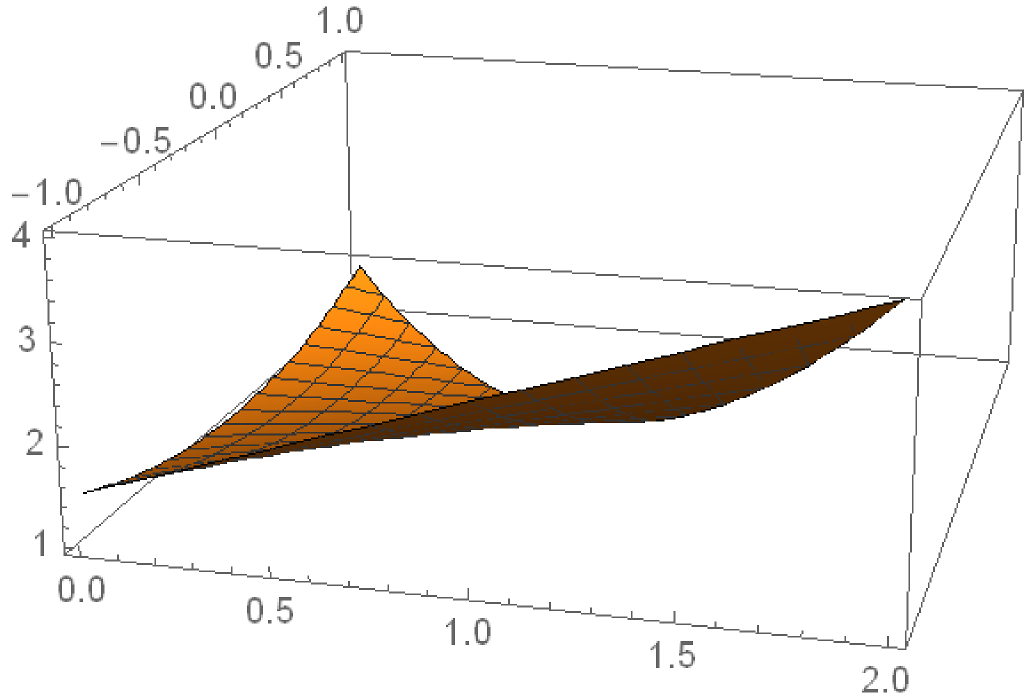
i) For each boundary, give an equation in terms of x and y. Find intersection with surface.

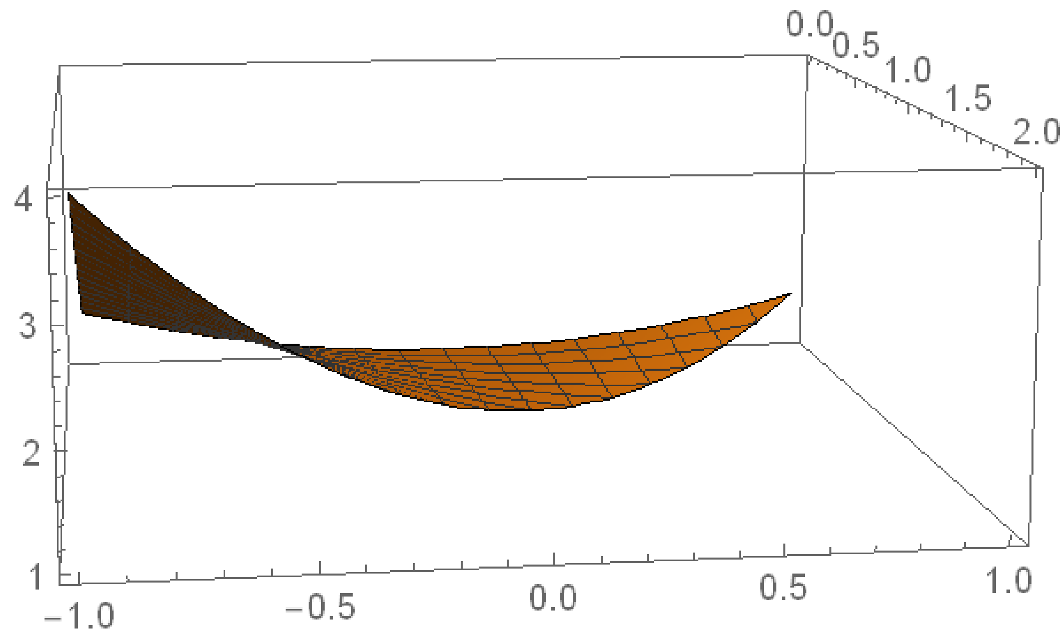
ii) Find critical numbers and endpoints for this one variable function. Label "corners".

Typical Example:

Let R be the triangular region in the xy-plane with corners at (0,-1), (0,1), and (2,-1). Above R, find the *absolute* (*global*) max and min of

$$f(x,y) = \frac{1}{4}x + \frac{1}{2}y^2 - xy + 1$$


1.0
0.5
0.0
−0.5
−1.0
4
3
2
1
0.0
0.5
1.0
1.5
2.0


0.0
0.5
1.0
1.5
2.0
4
3
2
1
−1.0
−0.5
0.0
0.5
1.0